Chapter: 5
Columns and struts:

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as column, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as strut i.e. connecting rods, piston rods etc.

What's the difference between a Strut and a Column?

1. Both Strut and Column are compression structural members.
2. Slenderness ratio of struts is high, whereas it is low for columns.
3. Struts fail due to buckling, but columns fail in compression.

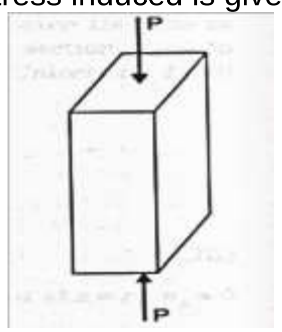
FAILURE OF A COLUMN:

The failure of a column takes place due to the anyone of the following stresses set up in the columns:

a) Direct compressive stresses.
b) Buckling stresses.
c) Combined of direct compressive and buckling stresses.

Failure of a Short Column:

A short column of uniform cross-sectional area A, subjected to an axial compressive load P, as shown in Fig. The compressive stress induced is given by; **$\sigma = P/A$**



If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

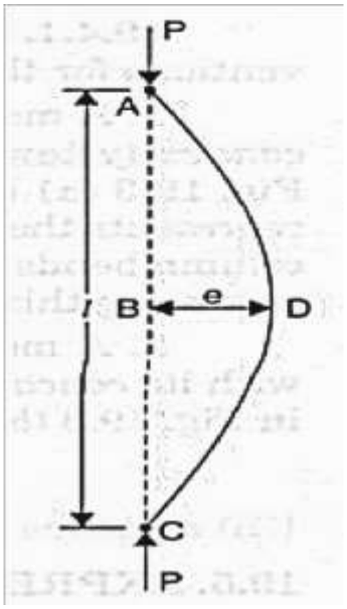
Let, P_c = Crushing load,
σ_c= Crushing stress, and
A = Area of cross-section

$\sigma_c = P_c/A$

All short columns fail due to crushing.

Failure of a Long Column:

A long column of uniform cross-sectional area A and of length l, subjected to an axial compressive load P, is shown in Fig. A column is known as long column, if the length of the column in comparison to its lateral dimensions, is very large Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown in figure. The buckling load at which the column just buckles, is known as buckling or crippling load. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.



Let l = Length of a long column
P = Load (compressive) at which the column has just buckled
A = Cross-sectional area of the column e = Maximum bending of the column at the centre

σ_o = Stress due to direct load = P/A
σ_b = Stress due to bending at the centre of the column = (P x e) /Z

Where,
Z = Section modulus about the axis of bending.

The extreme stresses on the mid-section are given by:
Maximum stress = $\sigma_o + \sigma_b$ and **Minimum stress = $\sigma_o - \sigma_b$**

The column will fail when maximum stress (i.e., $\sigma_o + \sigma_b$) is more than the crushing stress σ_c. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

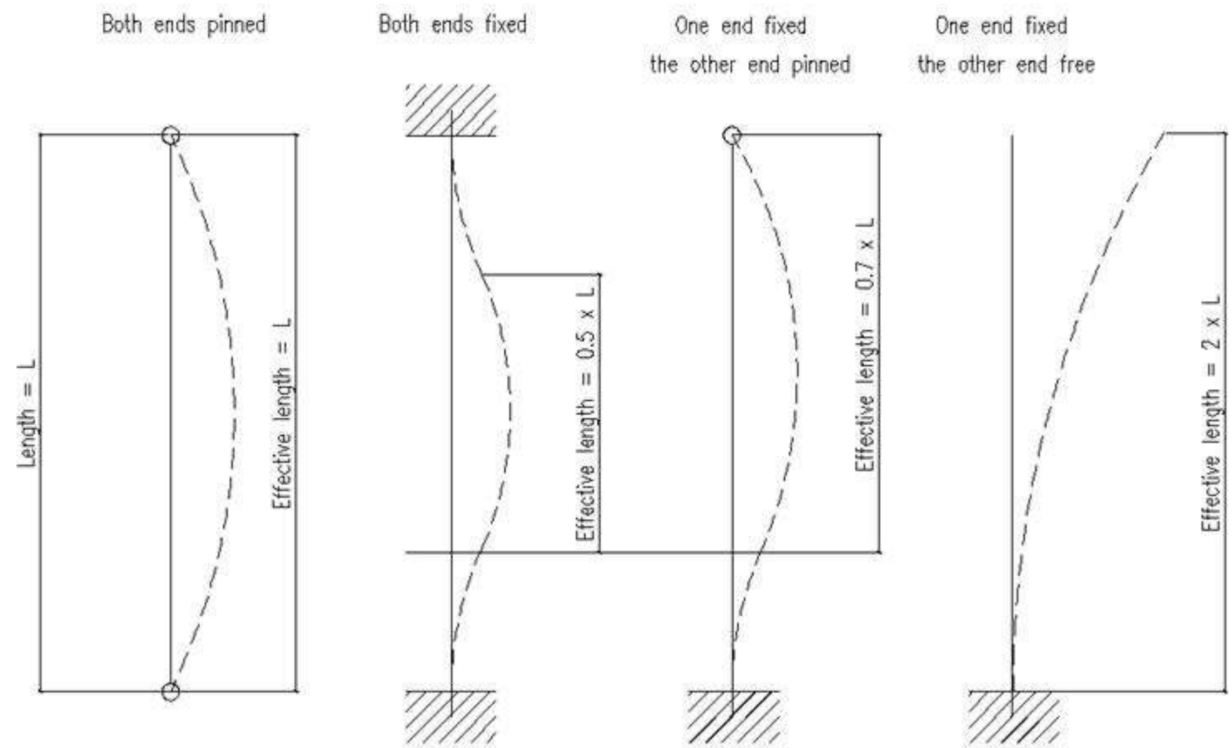
Assumptions made in the Euler's Column theory:

The following assumptions are made in the Euier's column theory:

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

End conditions for Long Columns: In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns takes place entirely due to buckling (or bending). The following four types of end conditions of the columns are important:

1. Both the ends of the column are hinged (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

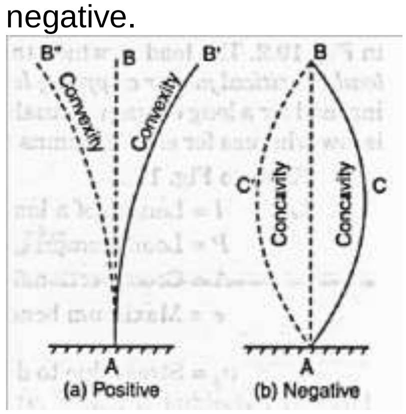


For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

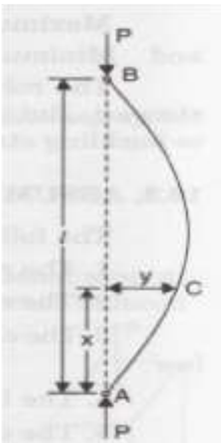
Sign Conventions:

The following sign conventions for the bending of the columns will be used:

1. A moment which will bend the column with its convexity towards its initial central line as shown in Fig. (a) is taken as positive. In Fig (a), AB represents the initial centre line of a column. Whether the column bends taking the shape AB' or AB", the moment producing this type of curvature is positive.
2. A moment which will tend to bend the column with its concavity towards its initial centre line as shown in Fig. (b) is taken as negative.



Expression for crippling load when both the ends of the Column are hinged: The load at which the column just buckles (or bends) is called crippling load. Consider a column AB of length l and uniform cross-sectional area, hinged at both of its ends A and B. Let P be the crippling load at which the column has just buckled. Due to the crippling load, the column wilI deflect into a curved form ACB as shown in Fig.



Consider any section at a distance r from the end A.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section = -P x y

(- ve sign is taken due to sign convention)

But moment $= EI \dfrac{d^2y}{dx^2}$.

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right)$$

Where C_1 and C_2 are the constants of integration and the values are obtained as follows:
At A, x = 0 and y = 0

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos 0^\circ + C_2 \sin 0$$
$$= C_1 \times 1 + C_2 \times 0 \qquad (\because \cos 0 = 1 \text{ and } \sin 0 = 0)$$
$$= C_1$$
$$\therefore \quad C_1 = 0. \qquad ...(ii)$$

(*ii*) At B, $x = l$ and $y = 0$ (See Fig. 19.4).

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right)$$
$$= 0 + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad [\because C_1 = 0 \text{ from equation } (ii)]$$
$$= C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right) \qquad ...(iii)$$

From equation (*iii*), it is clear that either $C_2 = 0$

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0.$$

As if $C_1 = 0$, then if C_2 is also equals to zero, then from Eq^n no. (i), we will find that y = 0. This means that the bending of the column will be zero or the column will not bend at all, which is not true.

$$\therefore \quad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } \ldots$$

$$\text{or} \quad l\sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \ldots$$

Taking the least practical value.

$$l\sqrt{\frac{P}{EI}} = \pi$$

$$P = \frac{\pi^2 EI}{l^2}.$$

Expression for crippling load when one end of the column is fixed and the other is free:

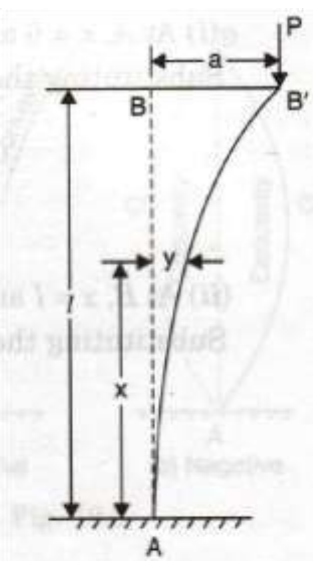
Consider a column AB, of length l and uniform cross-sectional area, fixed at the end A and free at the end B. The free end will sway sideways when load is applied at free end and curvature in the length l will be similar to that of upper half of the column whose both ends are hinged.

Let P is the crippling load at which the column has just buckled. Due to the crippling load P, the column will deflect as shown in Fig., in which AB is the original position of the column and AB', is the deflected position due to crippling load P.

Consider any section at a distance x from the fixed end A.

Let y = Deflection (or lateral displacement) at the section a = Deflection at the free end B'

Then moment at the section due to the crippling load = P (a - y)



+Ve sign is taken due to sign convention

$$\text{But moment is also} \quad = EI\frac{d^2y}{dx^2}$$

$\therefore$ Equating the two moments, we get

$$EI\frac{d^2y}{dx^2} = P(a - y) = P \cdot a - P \cdot y$$

$$\text{or} \quad EI\frac{d^2y}{dx^2} + P \cdot y = P \cdot a$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{P}{EI} \cdot a. \quad \ldots(A)$$

The solution of the Differential Equation is:

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) + a \quad \ldots(i)$$

Where C_1 and C_2 are the constants of integration and the values are obtained from the boundary conditions, which are as follows:

i. At fixed end, the deflection as well as slope will be zero.

Hence at end A (which is fixed), the deflection $y = 0$ and also slope $\frac{dy}{dx} = 0$.

Hence at A, $x = 0$ and $y = 0$

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos 0 + C_2 \sin 0 + a$$

$$= C_1 \times 1 + C_2 \times 0 + a \qquad (\because \cos 0 = 1, \sin 0 = 0)$$

$$= C_1 + a$$

$$\therefore \quad C_1 = -a \qquad \text{...(ii)}$$

At $A, x = 0$ and $\frac{dy}{dx} = 0.$

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1 \cdot (-1) \sin\left(x\sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 \cdot \sqrt{\frac{P}{EI}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sqrt{\frac{P}{EI}} \cos\left(x\sqrt{\frac{P}{EI}}\right)$$

But at A, $x = 0$ and $\frac{dy}{dx} = 0$.

$\therefore$ The above equation becomes as

$$0 = -C_1 \cdot \sqrt{\frac{P}{EI}} \sin 0 + C_2 \sqrt{\frac{P}{EI}} \cos 0$$

$$= -C_1 \sqrt{\frac{P}{EI}} \times 0 + C_2 \cdot \sqrt{\frac{P}{EI}} \times 1 = C_2 \sqrt{\frac{P}{EI}}.$$

From the above equation it is clear that either $C_2 = 0$.

$$\sqrt{\frac{P}{EI}} = 0.$$

But for the crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$$\therefore \quad C_2 = 0.$$

Substituting the values of $C_1 = -a$ and $C_2 = 0$ in equation (*i*), we get

$$y = -a \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + a. \qquad \text{...(iii)}$$

But at the free end of the column, x = l and y = a,

Substituting these values in equation (*iii*), we get

$$a = -a \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) + a$$

or $$0 = -a \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) \text{ or } a \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = 0$$

But 'a' cannot be equal to zero

$$\therefore \quad \cos\left(l.\sqrt{\frac{P}{EI}}\right) = 0 = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2} \text{ or } \cos\frac{5\pi}{2} \text{ or } ......$$

$$\therefore \quad l.\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \; ......$$

Taking the least practical value,

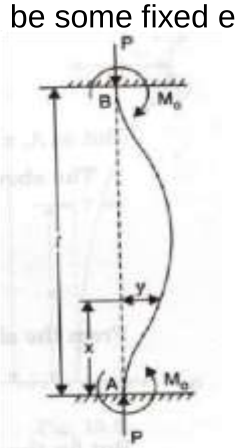
$$l.\sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{2l}$$

or $$P = \frac{\pi^2 EI}{4l^2}$$

Expression for the crippling load when both ends of the column are fixed:

Consider a column AB of length l and uniform cross-sectional area fixed at both ends A and B as shown in Fig. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown. Due to fixed ends, there will be fixed end moments say M_0 at the ends A and B. The fixed end moments will be acting in such direction so that slope at the fixed ends becomes zero.

Consider a section at a distance x from the end A. Let the deflection of the column at the section is y. As both the ends of the column are fixed and the column carries a crippling load, there will be some fixed end moments at A and B.



Then moment at the section = $M_0 - P.y$

But moment at the section is also = $EI\,\dfrac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI\,\frac{d^2y}{dx^2} = M_0 - P.y$$

or $$EI\,\frac{d^2y}{dx^2} + P\,.\,y = M_0$$

Let M_0 = Fixed end moments at A and B.

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{M_0}{EI} \quad ...(A)$$

$$= \frac{M_0}{EI} \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{M_0}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad ...(i)$$

where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. Boundary conditions are :

(i) At A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$ as A is a fixed end.

(ii) At B, $x = l$, $y = 0$ and also $\frac{dy}{dx} = 0$ as B is also a fixed end.

Substituting the value x = 0 and y = 0 in equation no (i), we get.

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{M_0}{P}$$
$$= C_1 + \frac{M_0}{P}$$
$$C_1 = -\frac{M_0}{P}$$

Differentiating equation (i), with respect to x, we get.

$$\frac{dy}{dx} = C_1 \cdot (-1) \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + 0$$
$$= -C_1 \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}}$$

Substituting the value $x = 0$ and $\frac{dy}{dx} = 0$, the above equation becomes

$$0 = -C_1 \times 0 + C_2 \times 1 \times \sqrt{\frac{P}{EI}} \quad (\because \sin 0 = 0 \text{ and } \cos 0 = 1)$$
$$= C_2 \sqrt{\frac{P}{EI}}$$

From the above equation, it is clear that either $C_2 = 0$ or $\sqrt{\frac{P}{EI}} = 0$. But for a given crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$\therefore$ $C_2 = 0$.

Now substituting the values of $C_1 = -\frac{M_0}{P}$ and $C_2 = 0$ in equation (i), we get

$$y = -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + 0 + \frac{M_0}{P}$$
$$= -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad ...(iii)$$

At the end B of the column, $x = l$ and $y = 0$.

Substituting these values in equation (iii), we get

$$0 = -\frac{M_0}{P} \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

or $$\frac{M_0}{P} \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}$$

or $$\cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P} \times \frac{P}{M_0} = 1 = \cos 0, \cos 2\pi, \cos 4\pi, \cos 6\pi, \ldots\ldots$$

$$\therefore \quad l \cdot \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, \ldots\ldots$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P = \frac{\pi^2 EI}{l^2}.$$

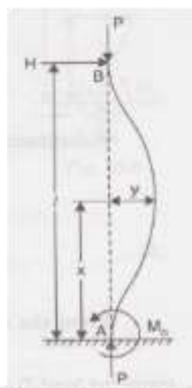
Expression for the crippling load when one end of the column is fixed and the other end is hinged (or pinned):

Consider a column AB of length l and uniform cross-sectional area, fixed at the end A and hinged at the end B as shown in Fig. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown in Fig. There will be fixed end moment (M_0) at the fixed end A. This will try to bring back the slope of deflected column zero at A. Hence it will be acting anticlockwise at A. The fixed end moment M_0 at A is to be balanced. This will be balanced by a horizontal reaction (H) at the top end B as shown in Fig.

Consider a section at a distance x from the end A

Let y = Deflection of the column at the section, M_0 = Fixed end moment at A, and

H = Horizontal reaction at B.



The moment at the section = Moment due to crippling load at B + Moment due to horizontal reaction at B

$$= -P \cdot y + H \cdot (l - x)$$

But the moment at the section is also

$$= EI \frac{d^2y}{dx^2}$$

Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = -P \cdot y + H(l - x)$$

or

$$EI \frac{d^2y}{dx^2} + P \cdot y = H(l - x)$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H}{EI}(l - x) \qquad \text{(Dividing by } EI\text{)} \quad \ldots(A)$$

$$= \frac{H}{EI}(l - x) \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{H(l - x)}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l-x) \qquad ...(i)$$

where C_1 and C_2 are constants of integration and their values are obtained from boundary conditions. Boundary conditions are :

(*i*) At the fixed end A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$

(*ii*) At the hinged end B, $x = l$ and $y = 0$.

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{H}{P}(l-0) = C_1 + \frac{H \cdot l}{P}$$

$$\therefore \quad C_1 = -\frac{H}{P} \cdot l \qquad ...(ii)$$

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1(-1)\sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$= -C_1 \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$.

$$\therefore \quad 0 = -C_1 \times 0 + C_2 \cdot 1 \cdot \sqrt{\frac{P}{EI}} - \frac{H}{P} \qquad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2\sqrt{\frac{P}{EI}} - \frac{H}{P} \quad \text{or} \quad C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$$

Substituting the values of $C_1 = -\frac{H}{P} \cdot l$ and $C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$ in equation (*i*), we get

$$y = -\frac{H}{P} \cdot l \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l-x)$$

At the end B, $x = l$ and $y = 0$.

Hence the above equation becomes as

$$0 = -\frac{H}{P}\, l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l-l)$$

$$= -\frac{H}{P}\, l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + 0$$

or $$\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\, l\cos\left(l\sqrt{\frac{P}{EI}}\right)$$

or $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P} \cdot l \times \frac{P}{H} \times \sqrt{\frac{P}{EI}} \cdot \cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$$= l \cdot \sqrt{\frac{P}{EI}} \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right)$$

or $$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l \cdot \sqrt{\frac{P}{EI}}$$

The solution to the above equation is, $l \cdot \sqrt{\frac{P}{EI}} = 4.5$ radians

Squaring both sides, we get

$$l^2 \cdot \frac{P}{EI} = 4.5^2 = 20.25$$

$$\therefore \quad P = 20.25\,\frac{EI}{l^2}$$

But approximately $20.25 = 2\pi^2$

$$\therefore \quad P = \frac{2\pi^2 EI}{l^2}$$

Effective length (or equivalent length) of a column:

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column
l = Actual length of the column and
P = Crippling load for the column

Then the crippling load for any type of end condition is given by, P = $\frac{\pi^2 EI}{L_e^2}$

The crippling load (P) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table below

S.No.	End conditions of column	Crippling load in terms of		Relation between effective length and actual length
		Actual length	Effective length	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia i.e., I_{xx} and I_{yy}. The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

Slenderness Ratio: The ratio of the actual length of the column to the least radius of gyration of the column is known as slenderness ratio. Sometimes the columns whose slenderness ratio is more than 80 are known as long column and those whose slenderness ratio is less than 80,is known as short column.

Limitations of the Euler's Formula:

if the slenderness ratio i.e. (l/k) is small the crippling stress (or the stress at failure) will be high. But for the column material the crippling-stress cannot be greater than the crushing stress. Hence when the slenderness ratio is less than a certain limit Euler's formula gives a value of crippling stress greater than the crushing stress. In the limiting case we can find the value of l/k, for which the crippling stress is equal to crushing stress.

Rankine's Formula:

We have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by Rankine is known as Rankine's formula, which is given as

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$$

Where, P = Crippling load by Rankine's formula

P_C = Crushing load = σ_c x A

σ_C = Ultimate crushing stress.

A = area of cross section.

P_E = Crippling load by Euler's formula = $\frac{\pi^2 EI}{L_e^2}$

For a given column material the crushing stress σ_c is a constant. Hence the crushing load P_c (which is equal to σ_c x A) will also be constant for a given cross-sectional area of the column. In above equation, P_c is constant and hence value of P depends upon the value of P_E. But for a given column material and given cross-sectional area, the value of P_E depends upon the effective length of the column.

(i) If the column is a short, which means the value of L_e is small, then the value of P_E will be large. Hence the value of $1/P_E$ will be small enough and is negligible as compared to the value of $1/P_C$. Neglecting the value of $1/P_E$ in equation (i), we get,

$$\frac{1}{P} \rightarrow \frac{1}{P_C} \text{ or } P \rightarrow P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. Also we have seen that short columns fail due to crushing.

(ii) If the column is long, which means the value of L_e is large. Then the value of P_E will be small and the value of $1/P_E$ will be large enough compared with $1/P_C$.Hence the value of $1/P_C$ may be neglected in equation (i).

$$\frac{1}{P} = \frac{1}{P_E} \text{ or } P \rightarrow P_E.$$

(iii) Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C \cdot P_E}$.

Now taking the reciprocals, P = $\frac{P_C P_E}{P_C+P_E} = \frac{P_C}{1+\frac{P_C}{P_E}} = \frac{\sigma_C.A}{1+\frac{\sigma_C.A}{\frac{\pi^2 EI}{L_e^2}}}$

But I = Ak^2; where k is the least radius of gyration.

P = $\frac{\sigma_C.A}{1+\frac{\sigma_C.AL_e^2}{\pi^2 EAK^2}} = \frac{\sigma_C.A}{1+\frac{\sigma_C.\ L_e^2}{\pi^2 E\ K^2}} = \frac{\sigma_C.A}{1+\frac{a.\ L_e^2}{K^2}}$; where a is called Rankine constant

S. No.	*Material*	σ_c *in N/mm²*	*a*
1.	Wrought Iron	250	$\frac{1}{9000}$
2.	Cast Iron	550	$\frac{1}{1600}$
3.	Mild Steel	320	$\frac{1}{7500}$
4.	Timber	50	$\frac{1}{750}$

Problem1. A hollow mild steel tube 6 m long 4 cm internal diameter and 6 mm thick is used as a strut with both ends hinged. Find the crippling load and safe load taking factor of safety as 3. Take E = 2 x 10^5 N / mm^2.

Problem2. A simply supported beam of length 4m is subjected to a uniformly distributed load of 30 KN/m over the whole span and deflects 15 mm at the centre. Determine the crippling load the beam is used as a column with the following conditions: (i) One end fixed and another end hinged (ii) Both the ends pin jointed.