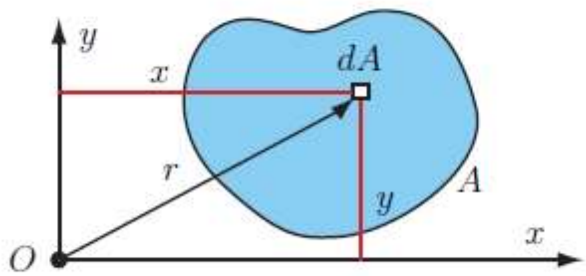
Chapter: 3

Moment of inertia:

The concept of inertia is provided by Newton's first law of motion. The property of matter by virtue of which it resists any change in its state of rest or of uniform motion is called inertia, the translator inertia is defined as mass whereas the rotational inertia is termed as moment of inertia. In other words, the moment of inertia is the rotational analogous of mass, i.e. it plays the role of resisting a change in rotational motion in quite the same sense as mass plays the role of resisting a change in translatory motion.

Moments of Inertia for Areas

The moment of inertia (second moment) of the area A about x and y axes, denoted as Ixx and Iyy, respectively, are

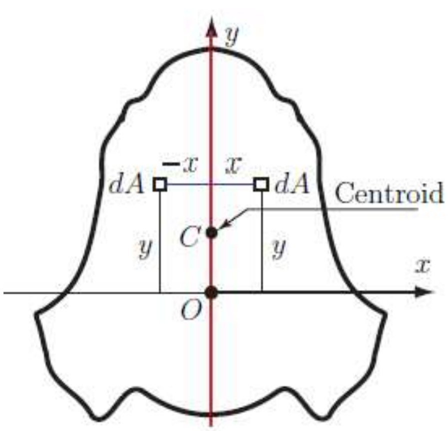


$$I_{xx} = \int_A y^2 \, dA,$$
$$I_{yy} = \int_A x^2 \, dA.$$

The second moment of area cannot be negative. The entire area may be concentrated at a single point (k_x;k_y) to give the same second moment of area for a given reference. The distances k_x and k_y are called the radii of gyration. Thus,

$$A k_x^2 = I_{xx} = \int_A y^2 \, dA \implies k_x^2 = \frac{\int_A y^2 \, dA}{A} = \frac{I_{xx}}{A}$$
$$A k_y^2 = I_{yy} = \int_A x^2 \, dA \implies k_y^2 = \frac{\int_A x^2 \, dA}{A} = \frac{I_{yy}}{A}$$

This point (k_x;k_y) depends on the shape of the area and on the position of the reference. The centroid location is independent of the reference position. The product of inertia for an area A is defined as, $I_{XY} = \int_A xy \, dA$



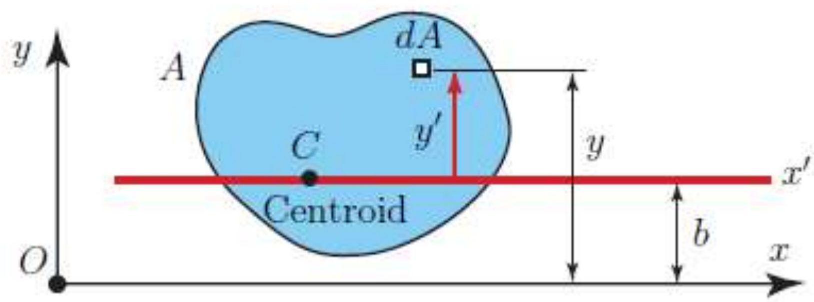
This quantity may be positive or negative and relates an area directly to a set of axes. If the area under consideration has an axis of symmetry, the product of area for this axis is zero. Consider the area in figure below, which is symmetrical about the vertical axis y. The planar Cartesian frame is xOy. The centroid is located somewhere along the symmetrical axis y. Two differential element of areas that are positioned as mirror images about the y axis are shown in figure. The contribution to the product of area of each elemental area is x.y.dA, but with opposite signs, and so the result is zero. The entire area is composed of such elemental area pairs, and the product of area is zero. The product of inertia for an area Ixy is zero (Ixy = 0) if either the x or y axis is an axis of symmetry for the area.

Parallel axis theorem:

Statement: the moment of inertia about any axis in the plane of the lamina equals to the sum of the moment of inertia about a parallel centroidal axis in the plane of lamina and the product of area of the lamina and the square of the distances between two axes.

The x axis in figure is parallel to an axis x-0 and it is at a distance b from the axis x'. The axis x' is going through the centroid C of the A area, and it is a centroidal axis. The second moment of area about the x axis is

$$I_{xx} = \int_A y^2 \, dA = \int_A (y' + b)^2 \, dA,$$



where the distance $y = y' + b$. Carrying out the operations

$$I_{xx} = \int_A y'^2 \, dA + 2b \int_A y' \, dA + A b^2.$$

The first term of the right-hand side is by definition $I_{x'x'}$,

$$I_{Cx'x'} = \int_A y'^2 \, dA.$$

The second term involves the first moment of area about the x' axis, and it is zero because the x' axis is a centroidal axis

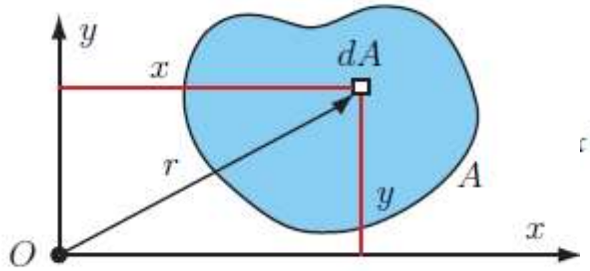
$$\int_A y' \, dA = 0.$$

The second moment of the area A about any axis I_{xx} is equal to the second moment of the area A about a parallel axis at centroid $I_{Cx'x'}$ plus $A b^2$, where b is the perpendicular distance between the axis for which the second moment is being computed and the parallel centroidal axis

$$I_{xx} = I_{Cx'x'} + A b^2.$$

Perpendicular axis theorem: if I_{XX} and I_{YY} be the moment of inertia of a lamina about mutually perpendicular axis Ox and Oy in the plane of lamina and I_{ZZ} be the moment of inertia about an axis normal to the lamina and passing through the point of intersection of the axis Ox and Oy, then $I_{ZZ} = I_{XX} + I_{XY}$

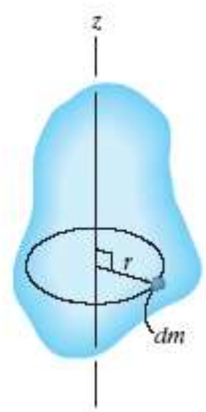
Polar Moment of Area: there is a reference xy associated with the origin O. Summing I_{xx} and I_{yy}



$$+ I_{yy} = \int_A y^2\,dA + \int_A x^2\,dA$$
$$= \int_A (x^2+y^2)\,dA = \int_A r^2\,dA,$$

where $r^2 = x^2+y^2$. The distance r^2 is independent of the orientation of the reference, and the sum $I_{xx} + I_{yy}$ is independent of the orientation of the coordinate system. Therefore, the sum of second moments of area about orthogonal axes is a function only of the position of the origin O for the axes.The polar moment of area about the origin O is, $I_O = I_{XX} + I_{YY}$

***Mass moment of inertia**:*

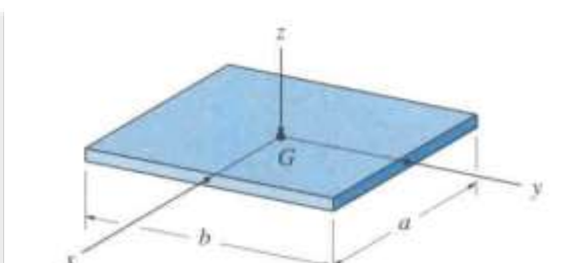


The mass moment of inertia is a measure of an object's resistance to rotation. Thus, the object's mass and how it is distributed both affect the mass moment of inertia. Mathematically, it is the integral

$$I = \int_m r^2\,dm = \int_V r^2\rho\,dV$$

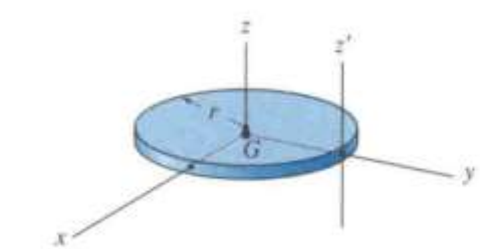
In this integral, r acts as the moment arm of the mass element and r is the density of the body. Thus, the value of I differs for each axis about which it is computed.

The figures below show the mass moment of inertia formulations for two flat plate shapes commonly used when working with three dimensional bodies. The shapes are often used as the differential element being integrated over the entire body.



Thin plate

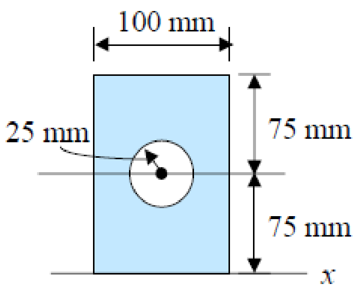
$I_{xx} = \frac{1}{12}mb^2 \quad I_{yy} = \frac{1}{12}ma^2 \quad I_{zz} = \frac{1}{12}m(a^2+b^2)$



Thin circular disk

$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$

Pbm1. Compute the moment of inertia of the composite area shown.

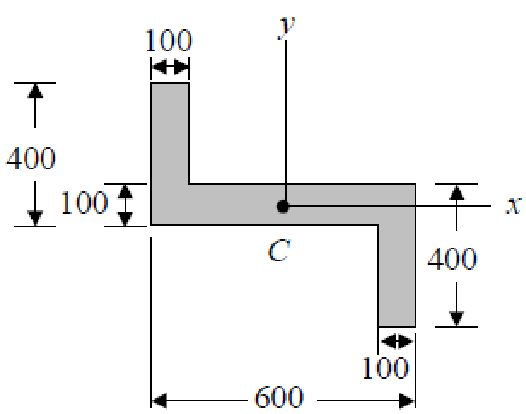


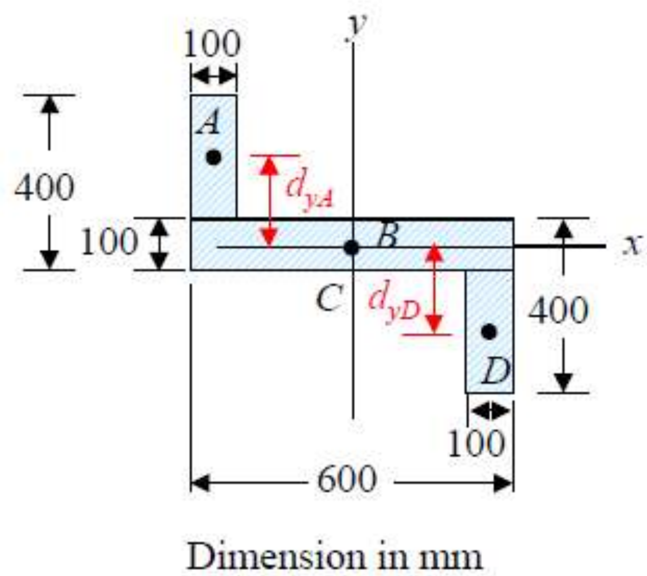
$$I_x = \left(\frac{bh^3}{3}\right)_{Rect} - (\bar{I}_x + A{d_y}^2)_{Cir}$$

$$= \left[\frac{1}{3}(100)(150)^3\right]_{Rect} - \left[\frac{1}{4}\pi(25)^4 + (\pi \times 25^2)(75)^2\right]_{Cir}$$

$$= 101\text{x}10^6 \text{ mm}^4$$

Pbm2. Determine the moments of inertia of the beam of cross-sectional area shown about the x and y centroidal axes.





$$I_x = (\bar{I}_x + A{d_y}^2)_A + (\bar{I}_x + A\overset{0}{{d_y}^2})_B + (\bar{I}_x + A{d_y}^2)_C$$

$$= \left[\frac{1}{12}(100)(300)^3 + (100 \times 300)(200)^2\right] + \left[\frac{1}{12}(600)(100)^3 + 0\right]$$

$$+ \left[\frac{1}{12}(100)(300)^3 + (100 \times 300)(200)^2\right]$$

$$= 2.9\text{x}10^9 \text{ mm}^4$$

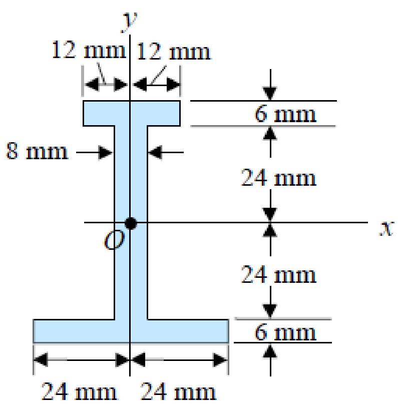
$$I_y = (\bar{I}_y + Ad_x^{\,2})_A + (\bar{I}_y + \overbrace{Ad_x^{\,2}}^{0})_B + (\bar{I}_y + Ad_x^{\,2})_C$$

$$= [\frac{1}{12}(300)(100)^3 + (100\times 300)(250)^2]_A + [\frac{1}{12}(100)(600)^3 + 0]_B$$

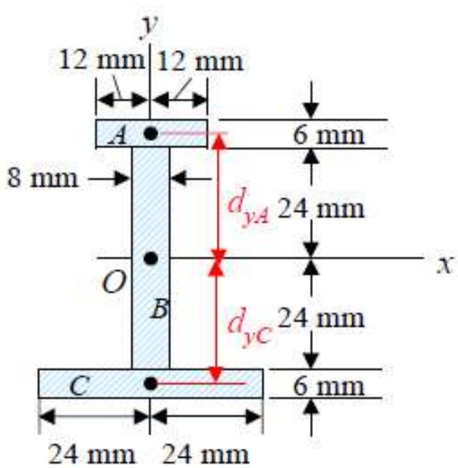
$$+ [\frac{1}{12}(300)(100)^3 + (100\times 300)(250)^2]_C$$

$$= 5.6\text{x}10^9\ \text{mm}^4$$

Pbm3. Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



SOLUTION



$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + \overbrace{Ad_y^{\,2}}^{0})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

$$= [\frac{1}{12}(24)(6)^3 + (24\times 6)(27)^2]_A$$

$$+ [\frac{1}{12}(8)(48)^3 + 0]_B$$

$$+ [\frac{1}{12}(48)(6)^3 + (48\times 6)(27)^2]_C$$

$$I_x = 390\text{x}10^3\ \text{mm}^4$$

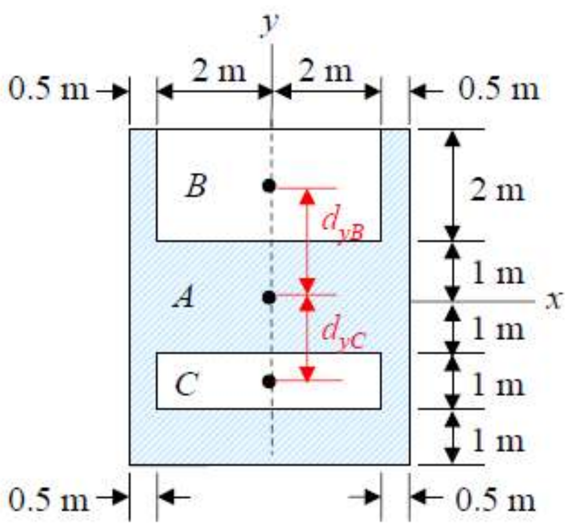
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times 10^3}{[(24\times 6) + (8\times 48) + (48\times 6)]}} = 21.9\ mm$$

$$I_y = (\bar{I}_y + \cancelto{0}{Ad_x^2})_A + (\bar{I}_y + \cancelto{0}{Ad_x^2})_B + (\bar{I}_y + \cancelto{0}{Ad_x^2})_C$$

$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

$I_y = 64.3\text{x}10^3 \text{ mm}^4$ ⇦ $\qquad k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times10^3}{[(24\times6)+(8\times48)+(48\times6)]}} = 8.87\ mm$ ⇦

Pbm 4. Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



$$I_x = (\bar{I}_x + \cancelto{0}{Ad_y^2})_{A5\times6} - (\bar{I}_x + Ad_y^2)_{B4\times2} - (\bar{I}_x + Ad_y^2)_{C4\times1}$$

$$= [\frac{1}{12}(5)(6)^3 + 0]_A - [\frac{1}{12}(4)(2)^3 + (2\times4)(2)^2]_B$$

$$-[\frac{1}{12}(4)(1)^3 + (4\times1)(1.5)^2]_C$$

$I_x = 46 \text{ m}^4$ ⇦

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.599\ m$$ ⇦

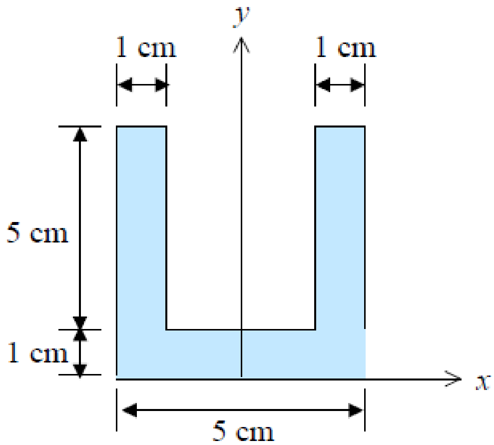
$$I_y = (\bar{I}_y + \cancelto{0}{Ad_x^2})_A - (\bar{I}_y + \cancelto{0}{Ad_x^2})_B - (\bar{I}_y + \cancelto{0}{Ad_x^2})_C$$

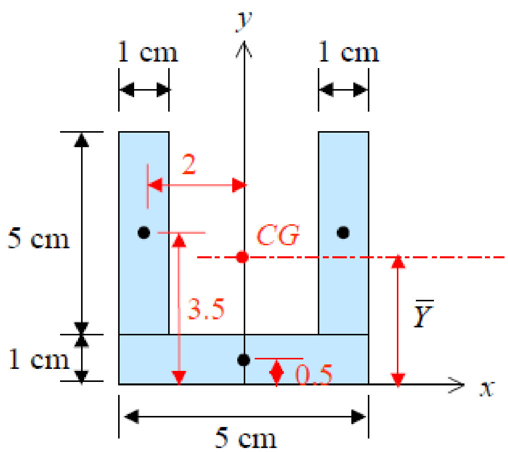
$$= [\frac{1}{12}(6)(5)^3]_A - [\frac{1}{12}(2)(4)^3]_B - [\frac{1}{12}(1)(4)^3]_C$$

$I_y = 46.5 \text{ m}^4$ ⇦

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.607\ m$$ ⇦

Pbm 5. Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes.





$$\bar{Y}\sum A = \sum \bar{y}A$$

$$\bar{Y} = \frac{2[(3.5)(5\times1)] + (0.5)(1\times5)}{3(5\times1)}$$

$$= 2.5\ cm$$

- **Moments of inertia about *x* axis**

$$I_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times1)(3.5)^2] + \frac{1}{3}(5)(1)^3$$

$$= 145\ cm^4$$

- **Moments of inertia about centroid**

$$\bar{I}_x = I_x - Ad_y{}^2$$

$$= 145 - (15)(2.5)^2$$

$$= 51.25\ cm^4$$

OR

$$\bar{I}_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times1)(1)^2]$$

$$+[(\frac{1}{12}(5)(1)^3 + (5\times1)(2)^2]$$

$$= 51.25\ cm^4$$

$$\bar{I}_y = I_y = 2[(\frac{1}{12}(5)(1)^3 + (5\times1)(2)^2] + \frac{1}{12}(1)(5)^3$$

$$= 51.25\ cm^4$$

$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848\ cm$$

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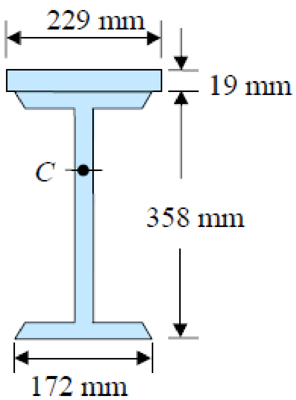
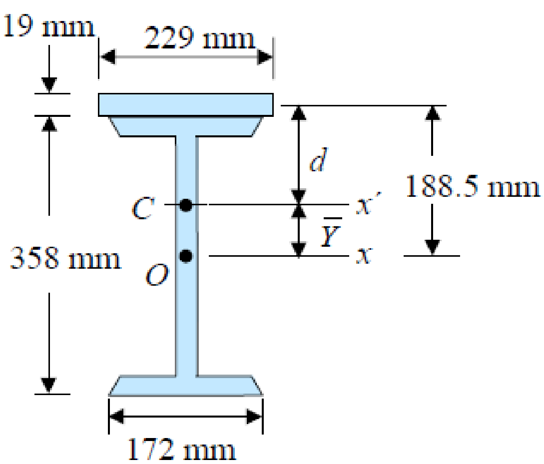
Pbm 6.. The strength of a W360 x 57 rolled-steel beam is increased by attaching a 229 mm x 19 mm plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid C of the section.



<u>SOLUTION</u>



- **Centroid**

The wide-flange shape of $W360$ x 57 found by referring to Fig. 9.13

$A = 7230\ \text{mm}^2 \qquad \bar{I}_x = 160.2\ mm^4$

$$A_{\text{plate}} = (229)(19) = 4351\ \text{mm}^2$$

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}(4351 + 7230) = (188.5)(4351) + (0)(7230)$$

$$\bar{Y} = 70.8\ mm$$

- **Moment of Inertia**

$$I_{x'} = (I_{x'})_{plate} + (I_{x'})_{wide-flange}$$

$$= (\bar{I}_{x'} + Ad^2)_{plate} + (\bar{I}_{x'} + A\bar{Y}^2)_{wide-flange}$$

$$= \left[\frac{1}{12}(229)(19)^3 + (4351)(188.5 - 70.8)^2\right]$$

$$+ \left[160.2\times10^6 + (7230)(70.8)^2\right]$$

$$= 256.8\times10^6\ mm^4$$

$$I_{x'} = 257\times10^6\ mm^4$$

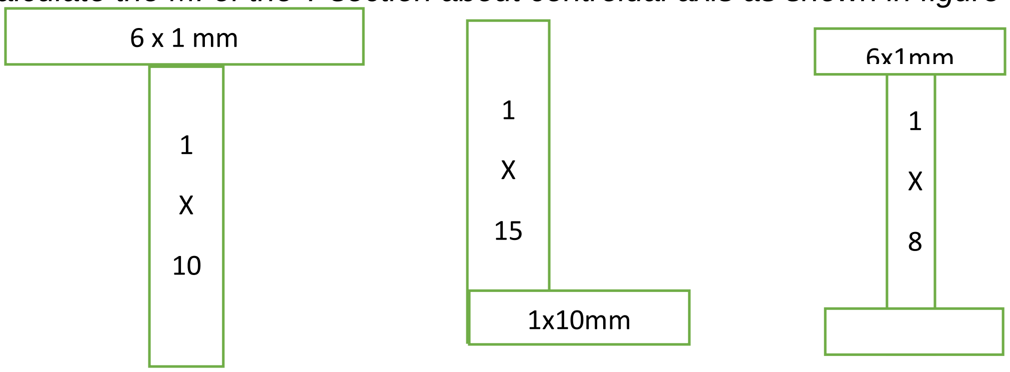
- **Radius of Gyration**

$$k_{x'}^2 = \frac{I_{x'}}{A} = \frac{256.8\times10^6}{(4351 + 7230)}$$

$$k_{x'} = 149\ mm$$

Exercise:

1. *Calculate the MI of the T section about centroidal axis as shown in figure below*



2. *Calculate the MI of the L section about centroidal axis as shown in figure above*
3. *Calculate the MI of the I section about centroidal axis as shown in figure above*
4. *Calculate the MI of the sections about centroidal axis as shown in figure below*